Efficient Numerical Methods for Fractional Laplacian and time fractional PDEs

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Two main difficulties with fractional PDEs:

- fractional derivatives are non-local operators which are much more difficult and expensive to deal with than local operators.
- fractional PDEs have weakly singularities at $t = 0$ and/or boundaries.

The following two situations will be considered:

- Part I. Solving the fractional Laplacian using the Caffarelli-Silverstre extension
- Part II. Space-time Petrov-Galerkin method for time-fractional diffusion equations
Part I: Fractional Laplacian equations in bounded domains

We consider the fractional Laplacian equation in a bounded domain $\Omega$:

\[
\begin{cases}
( (-\Delta)^s u(x) = f(x), & x = (x_1, x_2, \ldots, x_d) \in \Omega, \\
| u \rangle_{\partial \Omega} = 0,
\end{cases}
\]

where $0 < s < 1$, and the fractional Laplacian operator is defined through the spectral decomposition of Laplace operator.

Two review papers:
What is the fractional Laplacian? by Liske et al.
Numerical methods for fractional diffusion, by Bonito et al.

Three approaches:

- Using the discrete eigenfunctions of the Laplacian
- Using the Dunford-Taylor formula
  \[
  u = (-\Delta)^{-s} f = \frac{\sin s\pi}{\pi} \int_0^\infty \mu^{-s}(\mu I - \Delta)^{-1} f \, d\mu
  \]

- Using the Caffarelli-Silvestre extension (cf. Stinga & Torrea ’10)
Caffarelli-Silvestre extension

To overcome the difficulty associated with non-local operators, Caffarelli-Silvestre '07 (see Stinga & Torrea '10 for the bounded case) introduced an extension problem in $d + 1$ dimension with local differential operators:

$$
\begin{align*}
\nabla \cdot (y^\alpha \nabla U(x, y)) &= 0, & \text{in } \mathcal{D} = \Omega \times (0, \infty), \\
U(x, y) &= 0, & \text{on } \partial_L \mathcal{D} = \partial \Omega \times [0, \infty), \\
\lim_{y \to 0} y^\alpha U_y(x, y) &= -d_s f(x), & \lim_{y \to \infty} U(x, y) = 0
\end{align*}
$$

where $\alpha = 1 - 2s$ and $d_s = 2^{1-2s} \Gamma(1 - s)/\Gamma(s)$. Then, the solution of the fractional Laplacian equation can be expressed as

$$u(x) = U(x, 0).$$

Hence, one only needs to solve the above $d + 1$ dimensional problems with local differential operators.
Nochetto, Otarola & Salgado (2016) made a systematical study of the finite element approximation to the extension problem.

Fig. 2  Computational rate of convergence $\#(\mathcal{T}_Y)^{-s/(n+1)}$ for quasi-uniform meshes $\mathcal{T}_Y$, with $s = 0.2$ and $n = 1$.

Figure: $Q_1$ FEM convergence rate of the quasi-uniform mesh (in $y$), Nochetto et al 2016.
Q. Can we further improve the convergence rate in the extended direction?
The particular weight function $y^\alpha$ in the extension problem calls for the use of generalized Laguerre polynomials $\{L_k^\alpha(y)\}$ which are mutually orthogonal w.r.t. the weight $y^\alpha e^{-y/2}$.

Let us denote

$$Y_N^\alpha = \text{span}\{\hat{L}_k^\alpha(y) := L_k^\alpha(y)e^{-y/2}, k = 0, 1, \cdots, N\},$$

For the $x$-directions, one can use your favorite approximation space $X_K$, e.g., FEM or spectral method.

The Galerkin approximation for the extension problem is to find $u_{NK} \in X_{N,K} = Y_N^\alpha \times X_K$ such that

$$\left(y^\alpha \nabla u_{NK}, \nabla v\right)_D = d_s(f, v(x,0))_\Omega, \forall v \in X_{N,K}.$$
Fast solvers

- Let \( \{ \psi_j(x) \}_{1 \leq j \leq K} \) be a set of basis functions in \( X_K \), we write 
  \[ u_{NK} = \sum_{k=1}^{N} \sum_{j=1}^{K} \tilde{u}_{kj} \phi_k(y) \psi_j(x) \] 
  and \( U = (\tilde{u}_{kj}) \).

- Let us denote
  
  \[ S^y_{kj} = (y^\alpha \phi'_j(y), \phi'_k(y))(0, \infty), \quad M^y_{kj} = (y^\alpha \phi_j(y), \phi_k(y))(0, \infty), \]
  
  \[ S^x_{kj} = (\nabla_x \psi_j(x), \nabla_x \psi_k(x))_\Omega, \quad M^x_{kj} = (\psi_j(x), \psi_k(x))_\Omega. \]

  Then, the linear system for the Galerkin approximation is
  
  \[ S^y U M^x + M^y U S^x = F. \]

- Choice of basis functions for \( Y^\alpha_N \):
  Let \( \mathcal{L}_{-1}^\alpha(y) = 0 \), we set \( \phi_k(y) := \hat{\mathcal{L}}_k^\alpha y - \hat{\mathcal{L}}_k^\alpha(y) \). Then
  \( Y^\alpha_N = \text{span}\{ \phi_k(y) : k = 0, 1, \ldots, N \} \).

  and we have
  
  \[ \partial_y \phi_k(y) = \frac{1}{2} (\hat{\mathcal{L}}_{k-1}^\alpha y + \hat{\mathcal{L}}_k^\alpha(y)). \]

  Thanks to the orthogonality of generalized Laguerre functions, \( M^y \) and \( S^y \) are both symmetric penta-diagonal.
The above linear system can be solved efficiently by using the matrix-diagonalization method.

Let $S^y E = M^y E \Lambda$ where $(E, \Lambda)$ consists of eigenvectors and eigenvalues of $S^y \bar{x} = \lambda M^y \bar{x}$.

Setting the change of variable $U = EV$, we can reduce the matrix system to a sequence of $N$ problems in $x$-direction:

$$(\lambda_j M^x + S^x) \bar{v}_j = (E^t(M^y)^{-1} F)_j, \quad j = 1, 2, \cdots, N.$$ 

Since usually $N \ll K$, this procedure is very efficient, and is not intrusive as your favorite elliptic solver can be used.
Error estimates with generalized Laguerre functions:

$$\min_{v_N \in Y_N^\alpha} \| \hat{\partial}_y^l (u - v_N) \|_{Y^{\alpha+l}} \lesssim N^{(l-m)/2} \| \hat{\partial}_y^m u \|_{Y^{\alpha+m}}, \quad 0 \leq l \leq m$$

where $\hat{\partial}_y = (\partial_y + 1/2)$.

Then for the problem

$$-\partial_y (y^\alpha \partial_y u) = f, \quad u(0) = 0, \quad \lim_{y \to \infty} u(y) = 0,$$

the generalized Laguerre-Galerkin method in $Y_N^\alpha$ leads to:

$$\|(u - u_N)_y\|_{Y^\alpha} \lesssim N^{(1-m)/2} \| \hat{\partial}_y^m u \|_{Y^{\alpha+m-1}}.$$
The error estimate for Galerkin approximation of the extension problem in $X_{N,K}$ is:

$$\| U - U_{NK} \|_{1,y^\alpha} \lesssim N^{(1-m)/2} \| \hat{\partial}_y^m U \|_{y^{\alpha+m-1}}$$

\[ + \min_{v_K \in X_K} \| \nabla x (U(x,0) - v_K) \| \]

The first part is a typical result for spectral approximation.

Unfortunately, the solution is singular at $y = 0$ so that $\| \hat{\partial}_y^m u \|_{y^{\alpha+m-1}}$ is only bounded for $m = [2s] + 1$. So the Laguerre spectral method converges very slowly in the $y$-direction.
Figure: $\lambda = 2$, $s = 0.5, 0.31, 0.82$. 
Form of singularities at $t = 0$

A careful look at the extension problem reveals that the singularity can be explicitly identified so it is possible to use special basic functions to well represent the singular behavior at $y = 0$. By using a separation of variables approach, one finds that the solution to the extension problem can be expressed as

$$U(x, y) = \sum_{n=1}^{\infty} \tilde{U}_n \varphi_n(x) \psi_n(y)$$

where $\psi_n(y)$ is the solution of (Stinga & Torrea ’10)

$$\begin{cases} 
-\psi_n''(y) - \frac{1 - 2s}{y} \psi_n'(y) + \lambda_n \psi_n(y) = 0, & y \in \Lambda = (0, \infty), \\
\psi_n(0) = 1, \quad \lim_{y \to \infty} \psi_n(y) = 0,
\end{cases}$$

which can be expressed by Bessel function of the 2nd kind $K_s(z)$:

$$\psi_n(y) = c_s (\sqrt{\lambda_n y})^s K_s(\sqrt{\lambda_n y}), \quad c_s = 2^{1-s}/\Gamma(s).$$
We have

$$K_s(z) := \frac{s}{2} \frac{l_{-s}(z) - l_s(z)}{\sin(s\pi)}, \quad l_s(z) := \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(j + 1 + \alpha)} \left( \frac{z}{2} \right)^{2j+s}. $$

So we can derive

$$\psi_n(y) = cs(\sqrt{\lambda_n}y)^s K_s(\sqrt{\lambda_n}y)$$

$$= \frac{scs}{2\sin(s\pi)} \left\{ (\sqrt{\lambda_n}y)^s l_{-s}(\sqrt{\lambda_n}y) - (\sqrt{\lambda_n}y)^s l_s(\sqrt{\lambda_n}y) \right\} $$

$$= \frac{scs}{2\sin(s\pi)} \sum_{j=0}^{\infty} \frac{(\sqrt{\lambda_n}y)^{2j} - (\sqrt{\lambda_n}y)^{2j+2s}}{2^{2j+s} j!\Gamma(j + 2 - 2s)} $$

$$= g_{1,n}(y) + y^{2s} g_{2,n}(y),$$

where $g_{1,n}(y), \ g_{2,n}(y)$ are smooth functions.
Enriched spectral method

It is natural to add some singular parts to the approximation space in the $y$-direction:

$$Y_{N,k}^{\alpha} = Y_{N}^{\alpha} \oplus \{ y^{2s} \hat{\mathcal{L}}_{j}^{\alpha}(y) : j = 0, 1, \ldots, k \},$$

and the new approximation space for the extension problem is:

$$X_{N,K}^{k} = Y_{N}^{\alpha,k} \times X_{K}.$$ 

We have the following error estimate with the new approximation space:

$$\| u - U_{NK}^{k}(x, 0) \|_{H^{s}(\Omega)} \lesssim N^{-\left\lfloor \frac{2s}{2} \right\rfloor - k} + \min_{v_{K} \in X_{K}} \| \nabla(u - v_{K}) \|.$$
Solution of the linear system

One can apply the same matrix diagonalization process as before, but \((S_y, M_y)\) are usually severely ill conditioned since the added singular functions are "similar at \(y = 0\)" and have no orthogonal relation with the Laguerre functions. This approach can only be used for small \(k\) (which is usually enough).

**Figure:** Error behaviors with the enriched spectral method (1-D): \(s=0.2\)
$\mathbf{X}_h \times \mathbf{Y}_k^N$ such that 
\begin{equation}
\begin{aligned}
(y_1 - 2s \nabla U_h^N,k, \nabla V) D &= \int_{-1}^1 f(x) V(x,0) dx, \\
\forall V \in \mathbf{X}_h \times \mathbf{Y}_k^N.
\end{aligned}
\end{equation}

\[(4.26)\]

\textbf{Figure:} Smooth solution with a spectral method in $\Omega = (-1, 1)^2$: Left: $s = 0.2$, Right: $s = 0.8$. 

\textbf{Acknowledgment.} J.S. would like to thank Professor Ricardo Nochetto for stimulating discussions.
Figure: Non-smooth solution with a finite element method in $\Omega = (-1, 1)$: Left: $f(x) = 1, \ s = 0.2, \ \ \text{Right: } f(x) = (1 - x^2), \ s = 0.7.$
We consider the following class of fractional PDEs \(0 < \alpha < 1\):

\[
\frac{C_0 D_t^\alpha}{0} \nu(x, t) + \mathcal{L} \nu(x, t) + \mathcal{N}(\nu(x, t)) = 0, \quad \forall (x, t) \in \mathbb{D} := \Omega \times (0, T],
\]

with suitable boundary conditions and initial condition, where \(\mathcal{L}\) is a linear elliptic operator, \(\mathcal{N}\) is a lower-order nonlinear operator, and \(C_0 D_t^\alpha\) \((0 < \alpha < 1)\) is the left-sided Caputo fractional derivative of order \(\alpha\).

We can reformulate the above problem using the Riemann-Liouville derivative with homogeneous initial condition:

\[
\frac{0 D_t^\alpha}{0} u(x, t) + \mathcal{L} u(x, t) + \mathcal{N}(u(x, t)) = g(x, t), \quad \nu(x, 0) = 0.
\]
Two main difficulties in dealing with time-fractional PDEs:

- Solution at the next time step depends on solutions at all previous time steps.
- The solution is weakly singular at $t = 0$ so a usual approach will not lead to high accuracy.

Some existing approaches:

- Finite-difference methods with graded meshes at $t = 0$.
- Convolution integrals (Lubich '86, ...).
- Spectral-element method with geometric mesh leads exponential convergence (Mao & S. '17), but it is expensive and complicated.

Space-time spectral methods:

- using usual polynomials (Li & Xu '10) and Müntz polynomials (Hou & Xu '17);
We first consider the linear equations with $\mathcal{N} = 0$:

$$0D_t^\alpha u(x, t) + \mathcal{L} u(x, t) = g(x, t); \quad u(x, 0) = 0.$$ 

Petrov-Galerkin formulation: Find $u \in H_0^\alpha(I) \otimes H_L(\Omega)$ s.t.

$$\mathcal{A}(u, v) := (0D_t^\alpha u, v)_D + (\mathcal{L}^{\frac{1}{2}} u, \mathcal{L}^{\frac{1}{2}} v)_D = (g, v)_D, \quad \forall v \in L^2(I) \otimes H_L(\Omega),$$

where $H_L(\Omega) = \{ u \in L^2(\Omega) : (\mathcal{L}^{\frac{1}{2}} u, \mathcal{L}^{\frac{1}{2}} u) < \infty \}$.

The Petrov-Galerkin formulation is well-posed since

$$\mathcal{A}(u, 0D_t^\alpha u) = \|0D_t^\alpha u\|^2_{L^2(\mathbb{D})} + (\mathcal{L}^{\frac{1}{2}} u, 0D_t^\alpha \mathcal{L}^{\frac{1}{2}} u)_D \geq \|0D_t^\alpha u\|^2_{L^2(\mathbb{D})} + C_2(D_t^\alpha \mathcal{L}^{\frac{1}{2}} u, tD_{\mathbb{T}}^\alpha \mathcal{L}^{\frac{1}{2}} u)_D \geq C_2 \cos(\pi\alpha) \|0D_t^\alpha u\|^2_{L^2(I, H_L(\Omega))} \geq C_3(\|0D_t^\alpha u\|^2_{L^2(\mathbb{D})} + \|0D_t^\alpha u\|^2_{L^2(I, H_L(\Omega))}) := C_3 \|u\|^2_{B^\alpha(\mathbb{D})}.$$
We define shifted generalized Jacobi functions (or poly-fractonomials, Karniadakis & Zayernouri ’13)

\[ J_n^{(\alpha,\eta)}(t) = t^n \tilde{P}_n^{(\alpha,\eta)}(t), \quad t \in I, \quad n \geq 0, \]

where \( \tilde{P}_n^{(\alpha,\eta)}(t) = P_n^{(\alpha,\eta)}\left(\frac{2t - T}{T}\right) \) is the shifted Jacobi polynomial.

It satisfies the following remarkable property:

\[ 0D_t^\alpha J_n^{(-\alpha,\alpha)}(t) = \frac{\Gamma(n + \alpha + 1)}{n!} \tilde{P}_n^{(0,0)}(t). \]

So we define our approximation space in time by

\[ \mathcal{F}_N^{(\alpha)} := \{ t^\alpha \psi(t) : \psi(t) \in \mathcal{P}_N \} \]

\[ = \text{span}\{ J_n^{(-\alpha,\alpha)}(t) = t^\alpha \tilde{P}_n^{(-\alpha,\alpha)}(t) : 0 \leq n \leq N \}, \]

which incorporates the homogeneous boundary conditions at \( t = 0 \).
Let $V_h$ be a finite-dimensional approximation space of $V = H_L(\Omega)$:

$$V_h = \text{span}\{\phi_1, \phi_2, \cdots, \phi_M\}$$

Then, our Petrov-Galerkin method is: Find $u_L \in V_h \otimes F^\alpha_N$, such that

$$A(u_L, v_L) = (g, v_L)_D, \quad \forall v_L \in V_h \otimes P_N.$$ 

Q. The above linear system is of size $L = MN$. How to solve it efficiently?

A. Since the domain $D$ is a (separable) tensor product domain, we can employ a discrete separation of variables.
We write \( u_L(x, t) = \sum_{m=1}^M \sum_{n=0}^N \tilde{u}_{mn} \phi_m(x) J_n^{(-\alpha, \alpha)}(t), \) and denote

\[
    f_{mn} = \left( f, \phi_m(x) L_n^{(\alpha)}(t) \right)_\Omega, \quad F = (f_{mn}), \quad U = (\tilde{u}_h^m),
\]

\[
    s_{pq}^t = \int_0^1 D^\alpha_t J_q^{(-\alpha, \alpha)}(t)L_p(t)dt, \quad m_{pq}^t = \int_0^1 J_q^{(-\alpha, \alpha)}(t)L_p(t)dt,
\]

\[
    s_{pq}^h = \int_\Omega \mathcal{L}^{1/2} \phi_q \mathcal{L}^{1/2} \phi_p dx, \quad m_{pq}^h = \int_\Omega \phi_q \phi_p dx,
\]

\[
    S^t = (s_{pq}^t), \quad M^t = (m_{pq}^t), \quad S^h = (s_{pq}^h), \quad M^h = (m_{pq}^h).
\]

Then, we have

\[
    M^h U(S^t)^T + S^h U(M^t)^T = F.
\]

Note that \( S^t = I \), but \( M^t \) is full and non-symmetric.
Usual approach: diagonalization with eigen-decomposition

Let $E := (\bar{e}_0, \cdots, \bar{e}_N)$ be the matrix formed by the orthonormal eigenvectors of the generalized eigenvalue problem $M^t \bar{e}_j = \lambda_j S^t \bar{e}_j$ and $\Lambda = \text{diag}(\lambda_0, \cdots, \lambda_N)$, i.e.,

$$M^t E = S^t E \Lambda.$$

Setting $U = VE^T$, we arrive at

$$M^h V + S^h V \Lambda = G := F(S^t E)^{-T}.$$

Hence, the $n$-th column of the above matrix equation becomes:

$$(\lambda_n S^h + M^h)\mathbf{v}_n = \mathbf{g}_n, \quad 0 \leq n \leq N.$$

- Very efficient: only requires solving $N$ elliptic equations in $\Omega$.
- However, since $M^t$ is non-symmetric, this approach suffers from large roundoff errors.
Error comparison with eigen and QZ decompositions

Table: A comparison of decomposition errors between Eigen and QZ decompositions.

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We consider the following QZ decomposition:

\[ Q (S^t) T Z = A, \quad Q (M^t) T Z = B, \]

where \( Q, Z \) are unitary matrices, and \( A, B \) are upper triangular matrices.

Setting \( U = VQ \), we arrive at

\[ M^h V A + S^h V B = G := FZ. \]

We can solve the column vectors of \( V \) recursively,

\[ (a_{n,n} M^h + b_{n,n} S^h) v_n = g_n - h_{n-1}, \quad 0 \leq n \leq N. \]

where \( h_{n-1} = \sum_{k=0}^{n-1} (a_{k,n} M^h + b_{k,n} S^h) v_k \). with the total cost \( = O(N^2 M) + NT(M) (T(M) \text{ the cost of solving one elliptic equation}) \).
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**Lemma (Chen, S. & Wang ’16).** Let $\alpha \in (0, 1)$. Then, for any $v \in B_{-\alpha, \alpha}^s(I)$,

$$\|\pi_N^{(-\alpha, \alpha)} v - v\|_{\omega(-\alpha, \alpha)} \lesssim N^{-(\alpha + s)} \|D_t^{\alpha+s} v\|_{\omega(s, s)}.$$  

and

$$\|D_t^{\alpha} (\pi_N^{(-\alpha, \alpha)} v - v)\|_I \lesssim N^{-s} \|D_t^{\alpha+s} v\|_{\omega(s, s)}.$$ 

**Theorem.** If $u \in B^\alpha(\mathbb{D}) := H^s(I; L^2(\Omega)) \cap L^2(I; H^L(\Omega))$ and $0D_t^{\alpha+s} u \in L^2(\mathbb{D})$, we have

$$\|u - u_L\|_{B^\alpha(\mathbb{D})} \lesssim N^{-s} \|D_t^{\alpha+s} u\|_{L^2(\mathbb{D})} + \inf_{v_L(t, \cdot) \in V_h} \|u - v_L\|_{H^\alpha(I, H^L(\Omega))}.$$  

Unfortunately, $u$ has weak singularity at $t = 0$. The approximation space in time only includes the strongest singular term $t^\alpha$, so the achievable convergence rate in $N$ is limited.
Figure: Error in $B^\alpha$ against various $N$. Left: with the exact solution $u(x, y, t) = \sin(\pi x) \sin(\pi y) \cdot \sin(\pi t^\alpha)$ in $(-1, 1)^2$; Right: with $f(x, y, t) = (x^2 + y^2 - 1)$ in a disk.
We know from the Mittag-Leffler formula that the solution of fractional ODEs takes the form:

\[ u = \sum_{i,j}^{\infty} \gamma_{ij}^{\alpha} t^{i+j\alpha}. \]

The GJFs only include the singular terms \( t^{i+\alpha} \). In order to improve the convergence, we need to enrich the approximation space in time by other leading singular functions in the form of \( \{ t^{i+j\alpha} \} \):

\[ \mathcal{F}^{(k,\alpha)}_N(I) = \mathcal{F}^{(\alpha)}_N(I) \oplus \{ \text{first } k \text{ terms of } t^{i+j\alpha} \text{ not in } \mathcal{F}^{(\alpha)}_N(I) \}. \]
Then, the enriched Petrov-Galerkin method is: Find $u^k_L \in V_h \otimes \mathcal{F}_N^{(k,\alpha)}$, such that

$$A(u^k_L, v_L) = (g, v_L)_D, \quad \forall v_L \in V_h \otimes \mathcal{P}_{N+k}.$$ 

- Using a modified Gram-Schmidt process, one can construct an orthogonal set of $k$ enriched basis functions.
- The linear system can still be efficiently solved by using the QZ decomposition.
- The convergence rate can be increased to arbitrary order as we increase $k$. 
**Theorem.** Let \( \bar{k} + \nu \) \((0 < \nu < 1)\) be the first \( i + j\alpha \) not included in the enriched space.

- For \( \max\{0, \alpha - \frac{1}{2}\} < \nu < \alpha \),
  \[
  \| u - u^k_L \|_{B^{\alpha}(\Omega)} \lesssim N^{-\bar{k}} + \inf_{v_L(t, \cdot) \in V_h} \| u - v_L \|_{H^{\alpha}(I, H^L(\Omega))}. 
  \]

- For \( \alpha < \nu < \min\{1, \alpha + \frac{1}{2}\} \),
  \[
  \| u - u^k_L \|_{B^{\alpha}(\Omega)} \lesssim N^{-1-\bar{k}} + \inf_{v_L(t, \cdot) \in V_h} \| u - v_L \|_{H^{\alpha}(I, H^L(\Omega))}. 
  \]

- For \( \alpha + \frac{1}{2} < \nu < 1 \),
  \[
  \| u - u^k_L \|_{B^{\alpha}(\Omega)} \lesssim N^{-2-\bar{k}} + \inf_{v_L(t, \cdot) \in V_h} \| u - v_L \|_{H^{\alpha}(I, H^L(\Omega))}. 
  \]
Figure: Errors in $B^\alpha$ against various $N$ and $\alpha$. 
Extension to nonlinear problems

Consider now the nonlinear fractional PDEs:

\[ 0D_t^\alpha v(x, t) + \mathcal{L}v(x, t) + \mathcal{N}(v(x, t)) = g; \quad v(x, 0) = 0. \]

Let us denote

\[ \mathcal{A}(u, v) := (0D_t^\alpha u, v)_{\mathbb{D}} + (\mathcal{L}^{\frac{1}{2}} u, \mathcal{L}^{\frac{1}{2}} v)_{\mathbb{D}} + (\mathcal{N}(u), v)_{\mathbb{D}}. \]

**Petrov-Galerkin Approximation:** Find \( u_L \in V_h \otimes \mathcal{F}_N^{(\alpha)} \) s.t.

\[ \mathcal{A}(u_L, v_L) = (g, v_L)_{\mathbb{D}}, \quad \forall v_L \in V_h \otimes \mathcal{P}_N. \]

- The above nonlinear system can be solved by using Newton iteration which requires solving linear fractional PDEs with variable coefficients.
- We can use, as a preconditioner, the fast solver for linear fractional PDEs with constant coefficients. So the overall algorithm is still very efficient.
\[ \frac{C}{0}D_t^\alpha u(x, t) - \epsilon^2 \Delta u(x, t) + f(u(x, t)) = 0, \quad \forall (x, t) \in \Omega, \]

with the initial condition \( u_0(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ -1, & -1 \leq x < 0. \end{cases} \)

**Figure:** Solution profile. Left: \( \alpha = 0.7, \epsilon = 0.1 \) at various \( t \); Right: \( \epsilon = 0.1 \) at \( T = 1 \) with various \( \alpha \).
Part I. We developed efficient numerical methods for fractional Laplacian in bounded domains:

- we adopt the Caffarelli-Silverstre extension and developed efficient and accurate Laguerre-spectral method to deal with the singularity in the extended direction:
  - The method is not intrusive and can be applied to any discretization in space.
  - The method is much more efficient and easy to implement than using a finite-element approach in the extended direction.
  - The approach presented here can be extended to more general fractional elliptic equations.
Part II. We developed efficient space-time Petrov-Galerkin method for time fractional PDEs using the following two new approaches:

- We use the QZ decomposition which leads to accurate decompositions for non-symmetric matrices.
- We enrich the GJF approximation space by adding leading singular terms to resolve the weak singularity at \( t = 0 \).

Our Petrov-Galerkin method enjoys the following advantages:

- **Accuracy:** the enriched spectral method with a small number of modes can effectively resolve the weak singularity at \( t = 0 \).
- **Efficiency:** the total cost is dominated by a small number of elliptic solvers in space variables.
- **Flexibility:** one can use any Galerkin type discretization in space.
Some future directions:

- How to effectively deal with fractional Laplacian in integral form with the Caffarelli-Silvestre extension?

- The space-time Petrov-Galerkin method is only effective for simulation of short-times or smooth evolutions. How to develop an efficient space-time method with a spectral-element discretization in-time?

Thank you!